Math 132: Differential Topology

§ Exterior derivative

Let $\Omega^p(M)$ denote the set of all smooth p-forms on M.
(They're $\mathbb{R}$-vectorspaces, and in fact $C^\infty(M)$-modules, where $C^\infty(M) = \Omega^0(M)$)

Recall that we have the linear map

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M)$$
$$f \mapsto df.$$

This can be extended to arbitrary p-forms!

On an open subset U of $\mathbb{R}^m$, define the exterior derivative to be

$$\Omega^p(U) \xrightarrow{d} \Omega^{p+1}(U)$$
$$\omega = \sum_I a_I dx_I \mapsto d\omega = \sum_I da_I \wedge dx_I$$

Thm The exterior derivative d on smooth forms on $U \subset \mathbb{R}^m$ satisfies:

(a) (Linearity) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.

(b) (Multiplication law) $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^p \omega \wedge d\theta$ if ω is a p-form.

(c) (Cocycle condition) $d(d\omega) = 0$.

Furthermore, this is the unique operator with these properties that agrees with the previous definition of df for smooth functions f.

proof) Linearity is obvious, and the multiplication law is an easy computation.
The cocycle condition is also a straightforward computation:

$$\omega = \sum_I a_I dx_I \Rightarrow d\omega = \sum_I da_I \wedge dx_I = \sum_I \left(\sum_i \frac{\partial a_I}{\partial x_i} dx_i \right) \wedge dx_I$$

$$\Rightarrow d(d\omega) = \sum_I \left(\sum_i \sum_j \frac{\partial^2 a_I}{\partial x_i \partial x_j} dx_j \wedge dx_i \right) \wedge dx_I \underset{\uparrow}{=} 0.$$

(i,j)-term cancels with (j,i)-term

For uniqueness, if D were another operator satisfying (a), (b), (c) and $Df = df$,

then $D\omega = D\left(\sum_I a_I dx_I\right) \overset{(a),(b)}{=} \sum_I \left(Da_I \wedge dx_I + a_I D(dx_I) \right)$

$$\overset{(b)}{=} \sum_I \left(da_I \wedge dx_I + a_I \sum_{1 \le j \le p} (-1)^{j-1} dx_{i_1} \wedge \cdots \wedge D dx_{i_j} \wedge \cdots \wedge dx_{i_p} \right)$$

$$\overset{(c)}{=} \sum_I da_I \wedge dx_I = d\omega.$$

∎

<u>Cor</u> If $g : V \to U$ is a diffeomorphism of open subsets of $\mathbb{R}^m$,
then for every form ω on U, $d(g^*\omega) = g^*(d\omega)$.

proof) $D = (g^{-1})^* \circ d \circ g^*$ satisfies (a), (b), (c) and $D = d$ on functions.
Consequently, $D = d$ (i.e. $d \circ g^* = g^* \circ d$).

∎

• This last corollary allows us to define $\Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M)$ locally, using local parametrizations; the corollary tells us it is independent of the choice of local parametrizations.

The properties of exterior derivatives on Euclidean spaces carry over to arbitrary manifolds with boundary.

Moreover, d commutes with arbitrary pullback:

<u>Thm</u> If $g: N \to M$ is any smooth map of manifolds, then for any form ω on M, $d(g^*\omega) = g^*(d\omega)$.

proof) This holds for any 0-form (exercise), and also when $\omega = df$, for $d(g^* df) = d(dg^*f) = 0 = g^*(ddf)$.

Moreover, if this theorem holds for some ω and θ, then it does also for $\omega \wedge \theta$, for

$$d(g^*(\omega\wedge\theta)) = d(g^*\omega \wedge g^*\theta) = d(g^*\omega)\wedge g^*\theta + (-1)^p g^*\omega \wedge d(g^*\theta)$$

$$= g^* d\omega \wedge g^*\theta + (-1)^p g^*\omega \wedge g^* d\theta = g^* d(\omega\wedge\theta).$$

But every form on M is locally expressible as $\sum_I a_I dx_I$, and since the theorem is local, it holds for every form. ∎

<u>Ex</u> (Explicit calculation of d on $\mathbb{R}^3$)

$$\Omega^0(\mathbb{R}^3) \xrightarrow{d} \Omega^1(\mathbb{R}^3) \xrightarrow{d} \Omega^2(\mathbb{R}^3) \xrightarrow{d} \Omega^3(\mathbb{R}^3) \xrightarrow{d} 0$$

$$f \mapsto \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3$$

gradient!

$$f_1 dx_1 + f_2 dx_2 + f_3 dx_3 \mapsto \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dx_2 \wedge dx_3 + \text{cyc.}$$

curl!

$$f_1 dx_2\wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2 \mapsto \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3$$

divergence!

§ de Rham cohomology

So, on any smooth manifold M, we have a chain complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m(M) \to 0,$$

called the <u>de Rham complex</u>.

We say a p-form ω is <u>closed</u> if $d\omega = 0$ (i.e. if it is in $\mathrm{Ker}(d)$) and <u>exact</u> if $\omega = d\theta$ for some $(p-1)$-form θ (i.e. if it is in $\mathrm{Im}(d)$).

Since $d^2 = 0$, every exact form is closed, but the converse is not always true.

<u>Def</u> The p-th <u>de Rham cohomology group</u> of M, $H^p(M)$, is the $\mathbb{R}$-vector space of closed p-forms modulo exact p-forms,

i.e. $$H^p(M) = \frac{\mathrm{Ker}\left(\Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M)\right)}{\mathrm{Im}\left(\Omega^{p-1}(M) \xrightarrow{d} \Omega^p(M)\right)}.$$

<u>Rmk</u> Since d commutes with pullbacks, any smooth map $f: N \to M$ induces a linear map $f^*: H^p(M) \to H^p(N)$ on de Rham cohomology groups.

<u>Ex</u> A 0-form (i.e. a function) is closed iff it is constant on each component, so $\dim H^0(M) = \#$ connected components in M.

<u>Rmk</u> $$H^p(\mathbb{R}^m) = \begin{cases} \mathbb{R} & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}, \qquad \underset{(m>0)}{H^p(S^m)} = \begin{cases} \mathbb{R} & \text{if } p = 0 \text{ or } m \\ 0 & \text{otherwise} \end{cases}.$$